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ATTRITION PROCESSES WITH PARKING AREAS

Eleanor L. Schwartz

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Eleanor L. Schwartz

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INSTITUTE FOR DEFENSE ANALYSES

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PREFACE

This paper was prepared under the Institute for Defense Analyses' Central Research Program. It is an expansion and mathematical motivation of some concepts incorporated into several IDA-developed combat simulation models.

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ABSTRACT

This paper examines a class of combat processes in which targets can be located on "parking areas," so that an attack on a target can kill other targets on the same parking area. These processes have characteristics of both "point fire" and "area fire" models. A certain probabilistic model of an attack process is postulated; from it, exact and approximate expressions for expected numbers of targets killed are derived for a number of different sets of variations in the assumptions underlying the model. The paper explores in detail the relationship between these expressions and several previously-developed attrition equations in which targets were assumed not to be located on parking areas. The paper also provides rigorous mathematical justification for three equations that have been used in several combat simulations to compute attrition to aircraft on the ground. These equations are shown to be different special cases of the general model.

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A. INTRODUCTION

In aggregated, deterministic, large-scale models of combat (as opposed to fine-grained or Monte Carlo models), attrition to resources is often computed, implicitly or explicitly, by means of attrition equations. The results of such models are, in general, sensitive to the particular equations used. Accordingly, the study of attrition equations is an important aspect of large-scale modeling. Moreover, the realism of such models is increased if the equations used to compute attrition in a certain combat interaction take into account appropriate specific features of that interaction. If an attrition equation is derived mathematically from more fundamental assumptions about combat, and if those fundamental assumptions consider some appropriate features of combat, then the usefulness of such an equation will be enhanced, its properties and data requirements can be better understood, and the mathematical aspects of its suitability (or lack thereof) can be verified.

To take a specific example, an equation used to compute attrition to aircraft on an airbase (while on the ground) could consider that the aircraft might be parked in clusters ("aprons") and that an attack on an aircraft might kill some other aircraft in the same cluster.

Accordingly, this paper examines several stochastic attrition processes where targets can be located on "parking areas," in such a manner that an attack on a specific target might be able to kill other targets on the same parking area. Several attrition equations are derived; these represent exact or approximate formulas for the expected number of targets killed, under varying assumptions about the combat processes. Most of the attrition equations derived are of the "binomial" form; all can be considered discrete-time analogs of Lanchester equations, as discussed in References [5], [6], and [7].

One motivation for this paper is that three specific attrition equations with parking areas have been used in several combat simulation models to compute attrition to aircraft on airbases (caused by enemy aircraft). These equations, which were originally derived by L.B. Anderson, have been used in the IDATAM [3], NAVMOD [4], and OPTSA [2] models, among others. They differ in their assumptions on the choice of targets an attacker attacks and the specific targets that can be killed by an attack. (For any given run of the model, any one of these equations can be selected, by input, to calculate attrition.) In NAVMOD and IDATAM, these equations are used in a subroutine named ATRTAB (mnemonic for "attrition at airbase"), thus they will be referred to here as the "ATRTAB options." This paper develops generalizations of these equations and proves that they

correctly give the formulas for the expected numbers of targets killed under three different variations of the set of combat assumptions.¹ This paper also extends several previously-developed attrition equations to treat the case of parking areas.

The combat processes examined here are divided into three groups, based on the protocol by which a searcher chooses a target to attack. Section B looks at processes with "uniform allocation of fire," in which each searcher selects a target to attack uniformly from the targets it has detected. Several special cases are developed, including the second and third ATRTAB equations and generalizations of attrition equations discussed in References [9] and [11]. Section C considers a process with "strict priority allocation of fire," in which a searcher attacks the highest priority target it has detected (where priority is based on type of target). This work extends the combat process of Reference [8] to include parking areas, and the first ATRTAB equation falls out as a special case. Section D examines some combat processes in which a searcher detects whole parking areas, rather than individual targets.

This paper subsumes and supersedes Reference [13], which discussed some aspects of the combat process with parking areas and uniform allocation of fire.

B. ATTRITION PROCESSES WITH PARKING AREAS AND UNIFORM ALLOCATION OF FIRE

1. Basic Structure of the Processes

a. Terminology and Assumptions

The terminology introduced in this subsection and Subsection B.1.b will be used throughout the paper. Suppose that there are m types of searchers and n types of targets. Let $i=1,\dots,m$ index searcher types and $j=1,\dots,n$, target types. Let $\{1,\dots,n\}$ be partitioned into A sets U_1,\dots,U_A ; think of the index $a=1,\dots,A$ as denoting "type of parking area." The event that $j \in U_a$ (for some specific pair of j and a) should be interpreted as "targets of

¹ See Reference [4], Chapter IV, Section C.2.c, for more information on Subroutine ATRTAB. The current paper generalizes the central attrition calculations of Subroutine ATRTAB, but the subroutine also performs certain additional combat-related calculations not directly germane to this paper. It assumes only two types of targets: open (nonsheltered) aircraft and aircraft shelters, but some of the shelters might be occupied, and the subroutine makes additional calculations to determine a number of sheltered aircraft killed. Also, NAVMOD, IDATAM, and OPTSA model airbase attack as being directed against a set of identical "typical" airbases. Subroutine ATRTAB computes attrition for one such typical airbase; these results are then multiplied by the number of such airbases to compute overall attrition. See References [3] and [4] for details.

type j are located on parking areas of type a ." Since the sets U_a form a partition, each given type of target can be located on exactly one type of parking area. For each target type j , let $a(j)$ denote this parking area type; the distinction between the subscript a and the function $a(j)$ should be clear from context. Suppose that T_j targets of type j ($j=1,\dots,n$) and M_a parking areas of type a ($a=1,\dots,A$) are present. Then each type- a parking area is considered as having $t_j = T_j/M_a$ targets located on it, for each j in U_a (and as having no type- j targets located on it for all j not in U_a). For each value of j , $M_{a(j)}$ denotes the number of parking areas that can accommodate type- j targets, and $U_{a(j)}$ denotes the particular set (out of U_1,\dots,U_A) that contains j .¹

Suppose that there are S_i searchers of type i present ($i=1,\dots,m$). All searchers of any given searcher type and all targets of any given target type are assumed to be identical, and, as indicated above, all parking areas of any given parking area type are assumed to contain identical complements of targets. Consider an attrition process that proceeds according to the following assumptions.

- (1) At a fixed time, all targets become vulnerable to detection and attack.
- (2) Any particular searcher of type i detects any particular target of type j with probability d_{ij} .
- (3) Detections of different targets by a given searcher are mutually independent events.
- (4) The detection and attack processes of different searchers are mutually independent.
- (5) Of the targets it has detected, each searcher chooses one target according to a uniform distribution, and makes an (one) attack on the parking area containing that target. (A searcher that makes no detections makes no attack.)
- (6) If an attack by a type- i searcher is made on a given parking area, then each type- j target located on that parking area is killed with probability k_{ij} . The effects of different attacks on the same parking area are independent.

¹ The derivations in this paper are valid only if all S_i are nonnegative integers, all M_a are strictly positive integers, and T_j is a nonnegative integer multiple of $M_{a(j)}$ for each j (so that all t_j are nonnegative integers). The resulting equations, however, can be evaluated with any nonnegative (real) S_i and T_j and real $M_a \geq 1$, but are considered reasonable only if $M_{a(j)} \leq T_j$ for each j where T_j is nonzero. More research is needed to develop consistent and reasonable attrition equations that address the case in which this condition does not hold. Currently, Subroutine ATRTAB contains certain computations of a heuristic nature to treat this case. (The iterative use of attrition equations in a combat model can yield noninteger numbers of combatants.)

It is desired to find the expected number of type- j targets killed, for each j ($j=1,\dots,n$). The possibility that two or more searchers kill (i.e., lethally attack) the same target is explicitly considered.

Assumption (5) can be called the "uniform allocation of fire" rule for target choice. An assumption of this sort has been made in much of the previous work on binomial attrition processes (e.g., References [5], [6], [7], and [10]) and stands in contrast to the "strict priority allocation of fire" assumption of Reference [8] and Section C of the current paper and the "weighted allocation of fire" rule discussed in References [1] and [12].

Two points about Assumption (6) should be noted. First, it is possible for a target to be killed without having been detected--if some other target on the same parking area has been detected and that parking area has been attacked. Indeed, in some cases, it is possible for targets of type j to not be directly detectable--i.e., $d_{ij} = 0$ for that j and all i --yet still suffer attrition. Second, to compute the expected number of targets killed, no knowledge is necessary of the joint distribution of the targets killed on a parking area, given attack--the marginal probabilities of kill given attack k_{ij} suffice.

This combat process has elements of both "point fire" and "area fire" models. It is like point fire in that a searcher must make a detection in order to attack, but is like area fire in that one attack by one searcher can kill several targets, including targets not explicitly detected by that searcher.

At the outset, we emphasize that this paper does not derive an exact formula for the expected number of targets killed in this process. Note that in Assumption (2), the detection probabilities are a function of both searcher type and target type. To date, it has not been possible to derive succinct, exact expressions for expected numbers of targets killed in attrition processes where this condition holds and a uniform allocation of fire assumption is made (see References [5] and [9]). Section B.4 below develops several approximate formulas for the expected number of targets killed in the general process; Sections B.2 and B.3 derive exact closed-form expressions for the expected number of targets killed in several special cases. Most of these special cases involve some set of restrictions on the detection probabilities d_{ij} .

Suppose that each parking area contains exactly one target. Then Assumption (6) collapses into the regular assumption that an attack made on a target can kill only that target; i.e., the assumptions stated above become equivalent to those of the basic "no-parking-areas" attrition process of Chapter III of Reference [5]. No closed form expression for the

expected number of targets killed was derived for that process, and the expression that was derived is algebraically intractable. It is thus not surprising that no succinct closed form expression suggests itself for the expected number of targets killed in the general process with parking areas and uniform allocation of fire.

On the other hand, several approximate and special-case formulas have been developed in the no-parking-areas case for the expected number of targets killed (see References [7], [9], and [1]). In this paper, these formulas are extended to incorporate the feature of parking areas. Suppose that the parameters satisfy the assumptions:

$$\left. \begin{array}{ll} A=n & \\ a(j)=j & \text{for } j=1,\dots,n, \\ U_{a(j)}=\{j\} & \text{for } j=1,\dots,n, \text{ and} \\ M_{a(j)}=T_j & \text{for } j=1,\dots,n. \end{array} \right\} \quad (\text{B.1.1})$$

Then each parking area indeed does contain exactly one target,¹ and one would expect a formula for expected targets killed in a "parking areas" case to reduce to the corresponding "no-parking-areas" formula. This in fact does occur, and will be pointed out as appropriate.

b. Probabilistic Arguments

The definitions and probabilistic arguments presented in this section both indicate the difficulties of finding a general expression for the expected number of targets killed and can be used to derive many of the approximate and special-case expressions.

Let T_j^K denote the expected number of type- j targets killed. We first state a basic lemma, which is true by Assumptions (4) and (6) and standard methods (see Reference [11]).

Lemma 1:

$$T_j^K = T_j \left[1 - \prod_{i=1}^m (1-H_{ij})^{S_i} \right], \quad (\text{B.1.2})$$

where H_{ij} denotes the probability that a specific type- j target is killed by a specific type- i searcher.

¹ More generally, if the mapping a is any permutation of $\{1,\dots,n\}$, and $U_{a(j)} = \{j\}$ and $M_{a(j)} = T_j$ for each $j=1,\dots,n$, then each parking area contains exactly one target.

Let σ denote a specific searcher of type i , and τ , a specific target of type j . Suppose that target τ is located on parking area α (thus α is of type $a(j)$). Define the following events:

K -- σ kills τ (the probability $P(K)$ is equal to H_{ij} as defined above),

F_α -- σ attacks parking area α ,

G_a -- σ attacks a parking area of type a , i.e., σ makes an attack and the parking area σ attacks is of type a (defined for $a=1,\dots,A$), and

D -- σ detects at least one target (regardless of type).

From Assumption (6) it is clear that $P(K|F_\alpha) = k_{ij}$ and, of course $P(K|\bar{F}_\alpha)=0$ thus

$$H_{ij} \equiv P(K) = k_{ij} P(F_\alpha). \quad (B.1.3)$$

From Assumptions (2) and (3),

$$P(D) = 1 - \prod_{r=1}^n (1-d_{ir})^{T_r}. \quad (B.1.4)$$

By Assumption (5), searcher σ will attack no more than one parking area, and will attack a parking area if and only if it detects at least one target. Thus different events G_a are disjoint and

$$\bigcup_{a=1}^A G_a = D,$$

and thus

$$\sum_{a=1}^A P(G_a) = 1 - \prod_{r=1}^n (1-d_{ir})^{T_r}. \quad (B.1.5)$$

All parking areas of a given type are assumed to be identical, and at the outset, all targets of any given target type are equally detectable by searcher σ . Thus

$$P(F_\alpha | G_{a(j)}) = 1/M_{a(j)}. \quad (B.1.6)$$

(Of course, $P(F_\alpha | G_a) = 0$ for $a \neq a(j)$.) If the probabilities $P(G_a)$ could be obtained, then $P(K)$, i.e., H_{ij} , could be computed as

$$H_{ij} \equiv P(K) = k_{ij} P(G_{a(j)})/M_{a(j)}, \quad (B.1.7)$$

and the expected attrition would follow forthwith from Lemma 1.

The probabilities $P(G_\alpha)$ are not evident in the general case, but can be derived in several special cases.

Equation (B.1.3) reflects the fact that searcher σ kills target τ with (conditional) probability k_{ij} if σ attacks τ 's parking area (parking area α), whether or not τ is the "primary target" of α 's attack (in the sense that σ "chooses" τ in Assumption (5)). The idea of "primary target" can be used, however, to develop an alternative formula for $P(F_\alpha)$. The event that searcher σ attacks parking area α equals the event that some target on parking area α is the primary target of σ 's attack. Any given searcher will have at most one "primary target" in any realization of the combat process. All targets of a given type are assumed to be identical, and parking area α contains t_r (which equals $T_r/M_{\alpha(j)}$) targets of type r , for each r in $U_{\alpha(j)}$. All in all, then,

$$P(F_\alpha) = \sum_{r \in U_{\alpha(j)}} t_r c_{ir}, \quad (\text{B.1.8})$$

where c_{ir} is the probability that a specific type- r target is the primary target of searcher σ 's attack (recall that σ is of type i). (Strictly speaking, if $t_r = 0$, c_{ir} is undefined, but in this case interpret the product $t_r c_{ir}$ as zero.) In some special cases, as indicated below, it is possible and expeditious to compute the c_{ir} . Applying (B.1.8), (B.1.3) and Lemma 1 yields T_j^K .

2. The Case Where Nonzero Detection Probabilities Are a Function of Searcher Type Only

In this section it is assumed that the one-on-one detection probabilities d_{ij} are either zero or dependent only on the type (i) of searcher. More formally, the following assumption is made.

$$\left. \begin{array}{l} \text{For each searcher type } i \ (i=1, \dots, m) \text{ there exists} \\ \text{a value } \bar{d}_i \in [0, 1] \text{ and a subset } J_i \text{ of the set} \\ \text{of target types } \{1, \dots, n\} \text{ such that} \\ \text{--for all } j \in J_i, d_{ij} = \bar{d}_i, \text{ and} \\ \text{--for all } j \notin J_i, d_{ij} = 0. \end{array} \right\} \quad (\text{B.2.1})$$

Some of all of the sets J_i could be empty or equal to the whole space $\{1, \dots, n\}$. (The formulas below make sense if $\bar{d}_i = 0$ for some i , but the "natural" interpretation is that all \bar{d}_i are strictly positive.)

In this case, an exact and relatively concise expression for the expected number of targets killed can be derived, as will now be shown.

a. Derivation of Attrition Equation

Again, consider a specific searcher σ , of type i , and let D denote the event that σ detects at least one target. From Assumptions (2) and (3) and condition (B.2.1) it is clear that

$$P(D) = 1 - (1 - \bar{d}_i)^{\bar{V}_i}, \quad (\text{B.2.2})$$

where (the integer) \bar{V}_i is defined by

$$\bar{V}_i = \sum_{r \in J_i} T_r. \quad (\text{B.2.3})$$

One should think of \bar{V}_i as being the maximum number of targets that σ can detect. For now, assume that \bar{d}_i and \bar{V}_i are both strictly greater than zero. At the outset, all of these \bar{V}_i targets are equally detectable by searcher σ (because of condition (B.2.1)). By Assumption (5), the choice of primary target is made uniformly from the detected targets. Then by symmetry, the probabilities that any specific one of the \bar{V}_i targets detectable by σ becomes σ 's primary target are equal for all of the \bar{V}_i targets. Also by Assumption (5), searcher σ will choose exactly one primary target whenever σ detects at least one target. Thus the probability that any specific target detectable by σ is the primary target of σ is equal to

$$P(D)/\bar{V}_i. \quad (\text{B.2.4})$$

(This symmetry argument is similar to that of Reference [12]; expression (B.2.4) could also be derived by reasoning of the type given in Reference [11].)

Now consider a specific target τ , of type j , located on parking area α (which is thus of type $a(j)$), and let the events K and F_α and the probability $H_{ij} = P(K)$ be as described in Section B.1.b. To compute H_{ij} , first note that if any or all of the conditions $\bar{d}_i = 0$, $\bar{V}_i = 0$, and/or $J_i = \emptyset$ hold (\emptyset denotes the null set), searchers of type i can detect no targets at all, and $H_{ij} = 0$ (and $H_{ir} = 0$ for all target types r). Otherwise, H_{ij} can be computed from equations (B.1.3) and B.1.8); i.e., τ can be killed by σ precisely when one of the targets on parking area α is the primary target of σ 's attack. Parking area α contains $t_r (= T_r/M_{a(j)})$ targets of type r , for each $r \in U_{a(j)}$. Consider a target type $r \in U_{a(j)}$ such that $T_r > 0$, i.e., targets of type r are indeed present. If $r \notin J_i$, then searcher σ cannot detect type- r

targets, and thus a target of type r will never be the primary target of σ 's attack. If $r \in J_i$, then any particular type- r target will be the primary target of σ 's attack with probability $P(D)/\bar{V}_i$ (expression (B.2.4)). Substituting into equation (B.1.8) yields

$$P(F_\sigma) = \sum_{r \in U_{a(j)} \cap J_i} t_r P(D)/\bar{V}_i. \quad (B.2.5)$$

Combining the above results and applying (B.1.3) and Lemma 1 yields

$$T_j^K = T_j \left[1 - \prod_{i=1}^m (1 - H_{ij})^{S_i} \right], \quad (B.2.6)$$

where

$$H_{ij} = \begin{cases} 0, & \bar{V}_i = 0, \\ \frac{k_{ij}}{M_{a(j)}} \left(\sum_{r \in U_{a(j)} \cap J_i} T_r \bar{V}_i \right) \left[1 - (1 - \bar{d}_i)^{\bar{V}_i} \right], & \text{otherwise,} \end{cases} \quad (B.2.7)$$

where sums over the null set are considered to be zero.

Note that it is possible that for certain target types j , the detection probability d_{ij} might equal zero for all searcher types i (e.g., for all $j \notin J_i$), yet T_j^K as computed by (B.2.6) and (B.2.7) might be strictly greater than zero. This can be seen formally from equation (B.2.7): even if $j \notin J_i$, $U_{a(j)} \cap J_i$ could be nonempty and \bar{d}_i could be nonzero. That is, type- j targets, though not directly detectable, could be located on the same parking areas as detectable targets, and thus receive fire from an attacker, even though they could not be primary targets of an attacker.

If the parameters satisfy the conditions (B.1.1), i.e., each parking area contains exactly one target, it can be verified that T_j^K as computed by (B.2.6) and (B.2.7) reduces to equation (21) of Reference [9] (which is also equation (12) of Reference [11]). In this case, the property described in the preceding paragraph cannot occur, for under (B.1.1), $U_{a(j)} = \{j\}$ and thus in the expression for H_{ij} in (B.2.7) either the indicated sum or the term in brackets (or both) will be zero unless $d_{ij} > 0$.

b. A Further Special Case--All Detection Probabilities a Function of Searcher Type Only

Now consider the case where the detection probabilities satisfy a restricted version of the condition (B.2.1), such that for each searcher type i , J_i is the full set of target types $\{1, \dots, n\}$. Then, for each i , $d_{ij} = \bar{d}_i$ for all j . (For some i , \bar{d}_i may be zero; this implies that searchers of type i are completely ineffective.) In this case, the attrition equation given by (B.2.6) and (B.2.7) reduces to

$$T_j^K = T_j \left[1 - \prod_{i=1}^m \left(1 - \frac{k_{ij}}{TM_{a(j)}} \left(\sum_{r \in U_{a(j)}} T_r \right) \left[1 - (1 - \bar{d}_i)^T \right] \right)^{S_i} \right] \quad (B.2.8)$$

where $T = \sum_{r=1}^n T_r$ denotes the total number of targets.

As in the more general case of the preceding subsection, it is possible that for some j , $d_{ij} = 0$ for all i yet $T_j^K > 0$. If there is one target per parking area, i.e., the conditions (B.1.1) hold, then equation (B.2.8) reduces to the "basic heterogeneous binomial attrition equation," which has been used in a number of combat models (including References [2], [3], and [4]), and is derived and discussed in References [5], [6], [11], and [12].

3. Two Specific Attrition Processes With Parking Areas and Uniform Allocation of Fire

This section derives two attrition equations that are special cases of the general process described in Section B.1. The notation of Section B.1 is used throughout. Further restrictions on these special cases yield the second and third attrition equations available in (Subroutine ATRTAB of) the NAVMOD and IDATAM models to compute attrition to targets on an airbase caused by enemy attacking aircraft. Here, however, the special cases themselves will be referred to as "ATRTAB Option 2" and "ATRTAB Option 3," even though the equations are somewhat more general than the equations in the ATRTAB computer code.

The detection probabilities of these special cases need not satisfy the restrictions (B.2.1) of Section B.2. Even though there does not appear to be a simple attrition equation for the general process, in these special cases enough extra information is available to compute the probabilities $P(G_a)$ of equation (B.1.7) and thus to develop a formula for the expected number of targets killed.

For ease in presentation, the special cases are discussed here in a different order than they appear in Subroutine ATRTAB. The first special case discussed here corresponds to Option 3 of ATRTAB, and the second to Option 2. The first option of Subroutine ATRTAB involves priority, not uniform, allocation of fire, and is discussed in Section C.3, below.

a. ATRTAB Option 3

This special case assumes that there is just one type of parking area, i.e., $A=1$. Then the sum

$$\sum_{a=1}^A P(G_a)$$

which equals $P(D)$, consists of a single term $P(G_1)$, and $a(j) = 1$ for all j . Substituting in equation (B.1.7) yields, for all i and j ,

$$H_{ij} \equiv P(K) = k_{ij} P(D)/M_1. \quad (B.3.1)$$

Using expression (B.1.4) for $P(D)$ and substituting H_{ij} as above into equation (B.1.2) yields, for this special case,

$$T_j^K = T_j \left[1 - \prod_{i=1}^m \left(1 - \frac{k_{ij}}{M_1} \left[1 - \prod_{r=1}^n (1 - d_{ir}) T_r \right] \right)^{S_i} \right]. \quad (B.3.2)$$

Note that in equation (B.3.2), the detection probabilities appear only in a product, and if the term

$$P(D) = 1 - \prod_{r=1}^n (1 - d_{ir}) T_r \quad (B.3.3)$$

is nonzero, even if many of the d_{ir} are zero, then T_j^K can be nonzero for each j . That is, if a type- i searcher can detect at least one type of target, it can potentially kill a target of any type. For if there is only one type of parking area, then by the parameter assumptions, each parking area contains targets of all types.

Suppose that the conditions (B.1.1) hold, i.e., each parking area contains exactly one target, and also the assumption $A=1$ of ATRTAB Option 3 holds. Then $n=1$ --i.e., there is only one type of target--and equation (B.3.2) reduces to a special case of the "basic heterogeneous binomial attrition equation" (equation (3.14) of Reference [6]; see also the references listed in Section B.2.b, above) where there is only one type of target.

b. ATRTAB Option 2

This case arises if the detection probabilities d_{ij} are such that searchers of any given type can attack only one type of parking area. In general, one can consider, for each i , the set

$$J_i = \{j \mid d_{ij} > 0\}$$

(and j integer, $1 \leq j \leq n$) of target types that a type- i searcher can potentially detect. (This notation is slightly different from that of Section B.2.) ATRTAB Option 2 assumes that the detection probabilities d_{ij} satisfy the following condition:

$$\left. \begin{array}{l} \text{For each searcher type } i, \text{ there exists some} \\ \text{integer } b(i) \text{ (between 1 and } A, \text{ inclusive) such} \\ \text{that } J_i \subset U_{b(i)}. \end{array} \right\} \quad (\text{B.3.4})$$

Consider the example $m=2$, $n=6$, $A=3$, $U_1=\{1\}$, $U_2=\{2,3,4\}$, $U_3=\{5,6\}$, d_{12} , d_{14} , and d_{25} are strictly greater than zero, and all other d_{ij} equal zero. Then $J_1=\{2,4\}$, which is a subset of U_2 , and $J_2=\{5\}$, which is a subset of U_3 . Thus the detection probabilities of this example satisfy condition (B.3.4), with $b(1)=2$ and $b(2)=3$.^{1,2}

¹ Condition (B.3.4) also holds for the special case in which each J_i has one element, i.e., each type of searcher can potentially detect only one type of target.

² It is evident that if there is only one type of parking area ($A=1$) then the detection probabilities satisfy condition (B.3.4) with $b(i)=1$ for all i , since U_1 is the set of all target types. Thus ATRTAB Option 3 is a special case of ATRTAB Option 2. Because of its additional simplicity, ATRTAB Option 3 has been presented in a separate section. It can be verified that the formula (B.3.9) for expected attrition in ATRTAB Option 2 reduces to equation (B.3.2) if $A=1$.

The sets U_a form a partition, thus if condition (B.3.4) is satisfied, then $b(i)$ is unique for each i . This means that searchers of a given type can only detect--and thus (by Assumption (5)) attack--targets located on one type of parking area. The reverse need not be true: a given type of parking area might be attackable by several types of searchers--or no searchers at all. (In this latter case, all the targets located on such a parking area survive, as occurs with type-1 targets in the above example).

As always, the attrition to type- j targets can be computed by first computing, for each searcher type i , H_{ij} or $P(K)$, the probability that a specific type- i searcher kills a specific type- j target, and substituting in equation (B.1.2). Type- j targets are located on parking areas of type $a(j)$ and in ATRTAB Option 2, type- i searchers can only attack parking areas of type $b(i)$. As before, let σ be some specific type- i searcher and τ , some specific type- j target. The probabilities $P(G_a)$ that σ attacks some parking area of type a are zero except for $a = b(i)$. It is still true that

$$\sum_{a=1}^A P(G_a) = P(D) = 1 - \prod_{r=1}^n (1 - d_{ir})^{T_r},$$

thus

$$P(G_a) = \begin{cases} P(D) & a = b(i), \\ 0 & \text{otherwise.} \end{cases}$$

In particular

$$P(G_{a(j)}) = \begin{cases} P(D) & a(j) = b(i), \\ 0 & \text{otherwise;} \end{cases} \quad (\text{B.3.5})$$

substituting into equation (B.1.7) yields

$$H_{ij} \equiv P(K) = \begin{cases} k_{ij} [1 - \prod_{r=1}^n (1 - d_{ir})^{T_r}] / M_{a(j)} & b(i)=a(j) \\ 0 & \text{otherwise.} \end{cases} \quad (\text{B.3.6})$$

The definition of $b(i)$ implies that

$$b(i) = a(j) \quad \text{iff} \quad J_i \subset U_{a(j)}. \quad (\text{B.3.7})$$

For each j , define the subset of $\{1, \dots, m\}$

$$C_j = \{i \mid J_i \subset U_{a(j)}\}. \quad (\text{B.3.8})$$

Given that condition (B.3.4) holds, C_j is the set of types of searchers that can attack type- $a(j)$ parking areas. It is possible that C_j is the null set for some j --this occurs if all the types of targets located on type- $a(j)$ parking areas are undetectable by searchers of any type. Substituting the H_{ij} as defined above into equation (B.1.2) and simplifying yields, for each j in turn,

$$T_j^K = T_j \left[1 - \prod_{i \in C_j} \left(1 - \frac{k_{ij}}{M_{a(j)}} \left[1 - \prod_{r=1}^n (1 - d_{ir})^{T_r} \right] \right)^{S_i} \right], \quad (\text{B.3.9})$$

where a product taken over the null set is interpreted as unity.

It is interesting to note that if condition (B.3.4) holds, then (for each j) the set C_j defined above always contains the set

$$I_j = \{i \mid d_{ij} > 0\} = \{i \mid j \in J_i\} \quad (\text{B.3.10})$$

but is not necessarily equal to I_j . (I_j is the set of types of searchers that can detect type- j targets.) In the example mentioned earlier, I_3 is the null set, but $C_3 = \{1\}$ --and note that type-3 targets, by being co-located with type-2 and type-4 targets, are potentially vulnerable to type-1 searchers, though not directly detectable by them.

The implementation of ATRTAB Option 2 in the combat models of References [2], [3], and [4] at first glance does not appear similar to the description above. In that implementation, the attacker's sorties are preallocated between attacking open aircraft only or aircraft shelters only. A sortie preallocated to attack open aircraft only is not allowed to attack (or kill) a shelter, even if it detects some shelters but no open aircraft, and vice versa. In the process of Section B.1 (of the current paper), Assumption (5) implies that a searcher is capable of potentially attacking any target it detects, regardless of type. If different types of targets are located on different types of parking areas, however, preallocation can be modeled in this process by considering searchers preallocated to different target types as different types (i) of searchers and by setting, for each i , $d_{ij} = 0$ for all target types j except the one type $j(i)$ to which type- i searchers are preallocated. Then only targets of type $j(i)$ can be the primary targets of a type- i searcher's attack, and no targets of other types can be incidentally killed by such a searcher. Furthermore, each set J_i then consists of the single element $j(i)$, thus condition (B.3.4) is satisfied and equation (B.3.9), i.e., ATRTAB Option 2 as derived in the current paper, can be used to compute attrition. In the implementation of this option in the models of References [2], [3], and [4], open aircraft and aircraft shelters

are indeed assumed to be located on different types of parking areas (and in addition, the number of "parking areas for shelters" equals the number of shelters, so that an attack on a shelter can kill only that shelter). (It is clear that several generalizations of this structure might be possible--e.g., requiring the preallocation of primary target types but allowing incidental kills of targets of other types.)

4. Toward a General Attrition Equation

All the attrition equations presented so far have been exact expressions for the expected number of targets killed, i.e., they can be rigorously derived from the assumptions of the combat process. In contrast, this section presents and gives justification for three attrition equations which are approximate expressions for the expected number of targets killed. These expressions accept general d_{ij} and have concise forms. Subsection a extends some concepts of Reference [9] to the parking areas case, developing two equations that contain exponential terms. Subsection b presents an equation of "binomial" form that reduces to the appropriate exact expression in each of the special cases of Sections B.2, B.3.a, and B.3.b, above.

a. Use of Poisson Approximations

This section develops an approximate formula for the probability c_{ir} that a specific type- r target (target ρ) is the primary target of a specific type- i searcher (searcher σ). Equations (B.1.8), (B.1.3), and (B.1.2) (Lemma 1) can then be applied to yield an approximate expression for the expected number of type- j targets killed, T_j^K , for each j in turn.

The following arguments are essentially identical to those of Reference [9], with different notation. Define the following random variables:

L_r = number of targets of type r , other than target ρ , that searcher σ detects,

L_s = number of targets of type s that searcher σ detects (each subscript $s \neq r$ defines a different random variable), and

$$L = \sum_{s=1}^n L_s = \text{total number of targets, other than target } \rho, \text{ that searcher } \sigma \text{ detects.}$$

By Assumptions (2) and (3),

- L_r follows a binomial distribution with parameters $T_r - 1$ and d_{ir} ,
- L_s (for each $s \neq r$) follows a binomial distribution with parameters T_s and d_{is} , and
- All random variables L_r and L_s are mutually independent.

Also, searcher σ detects target ρ with probability d_{ir} , and by Assumption (5)

$$c_{ir} = \sum_{l=0}^{T-1} \frac{1}{l+1} d_{ir} P(L=l), \quad (\text{B.4.1})$$

where T denotes the total number of targets.¹ Unfortunately, since the probabilities d_{is} are not necessarily the same for different s , the probability distribution of L does not have a simple form. Use of the Poisson approximation to the binomial, however, yields the following results:

- L_r is approximately distributed Poisson with mean $d_{ir}(T_r - 1)$,
- L_s is approximately distributed Poisson with mean $d_{is}T_s$ (for each $s=1, \dots, n; s \neq r$), and
- L is approximately distributed Poisson with mean

$$\mu_{ir} = d_{ir}(T_r - 1) + \sum_{s \neq r} d_{is}T_s. \quad (\text{B.4.2})$$

Therefore

$$c_{ir} = \sum_{l=0}^{\infty} \frac{1}{l+1} d_{ir} \frac{(\mu_{ir})^l e^{-\mu_{ir}}}{l!} = \frac{d_{ir}}{\mu_{ir}} (1 - e^{-\mu_{ir}}). \quad (\text{B.4.3})$$

(It is conceivable that $\mu_{ir}=0$. Given that all parameters are nonnegative and that the numbers of searchers and targets are integers, this implies that target ρ is the only type- r target present and that any other targets present are undetectable by searcher σ . Thus, if σ detects ρ , ρ will be the primary target of σ 's attack, thus $c_{ir}=d_{ir}$. This is consistent with the interpretation of $(1-e^{-\mu_{ir}})/\mu_{ir} = 1$ at $\mu_{ir} = 0$ via L'Hopital's rule.)

Applying equations (B.1.8) and (B.1.3) and Lemma 1 yields

¹ Throughout this paper, the symbol "l" is to be read as a lower case "ell."

$$T_j^K \approx T_j \left[1 - \prod_{i=1}^m \left(1 - \frac{k_{ij}}{M_{a(j)}} \sum_{r \in U_{a(j)}} \frac{d_{ir} T_r}{\mu_{ir}} (1 - e^{-\mu_{ir}}) \right)^{S_i} \right], \quad (B.4.4)$$

where the μ_{ir} are defined by equation (B.4.2) and $(1 - e^{-0})/0$ is interpreted as 1, as just explained.

If the conditions (B.1.1) hold, i.e., each parking area contains exactly one target, then equation (B.4.4) reduces to equation (13) of Reference [9]. Note that Reference [9] derives an error bound for the approximation of its equation (13) to the exact equation. This derivation could probably be extended to the parking areas case, but this has not been done here (and the resulting error bound might be large).

A somewhat simpler approximate attrition equation can be developed by replacing each μ_{ij} with

$$\bar{\mu}_i = \sum_{s=1}^n d_{is} T_s, \quad (B.4.5)$$

yielding

$$T_j^K \approx T_j \left[1 - \prod_{i=1}^m \left(1 - \frac{k_{ij} (1 - e^{-\bar{\mu}_i})}{M_{a(j)} \bar{\mu}_i} \sum_{r \in U_{a(j)}} d_{ir} T_r \right)^{S_i} \right], \quad (B.4.6)$$

where, again, $(1 - e^{-0})/0$ is interpreted as 1 (although $\bar{\mu}_i = 0$ implies that type- i searchers are completely ineffective, and if $\bar{\mu}_i = 0$, the indicated summation in (B.4.6) is also zero and the i th product term is 1). If the "no-parking-areas" conditions (B.1.1) hold, (B.4.6) reduces to equation (14) of Reference [9]. Reference [9] derives an error bound for the approximation of its equation (14) to the exact case; the derivation could probably be extended to the parking areas case, but this has not been performed here.

b. An Approximate General Attrition Equation of Binomial Form

To round out the section on attrition processes with parking areas and uniform allocation of fire, we present an approximate equation for the expected number of targets of type j killed which:

- does not contain exponential terms,
- accepts general detection probabilities d_{ij} , functions of both searcher type and target type,
- is relatively simple to evaluate, and
- is exact for all the special cases discussed in Sections B.2 and B.3.

To develop this attrition equation, first note that, for each i and r ,

$$e^{-d_{ir}T_r} \approx (1 - d_{ir})^{T_r}, \quad (\text{B.4.7})$$

thus

$$e^{-\bar{\mu}_i} \approx \prod_{r=1}^n (1 - d_{ir})^{T_r}, \quad (\text{B.4.8})$$

where $\bar{\mu}_i$ is defined by equation (B.4.5). One can "unexponentiate" the Poisson approximation by substituting (B.4.8) and (B.4.5) in (B.4.6). This yields the equation

$$T_j^K \approx T_j \left[1 - \prod_{i=1}^m \left(1 - \frac{k_{ij}}{M_{a(j)}} \left[\frac{\sum_{r \in U_{a(j)}} d_{ir} T_r}{\sum_{r=1}^n d_{ir} T_r} \right] \left[1 - \prod_{r=1}^n (1 - d_{ir})^{T_r} \right] \right)^{S_i} \right] \quad (\text{B.4.9})$$

If there is an i such that

$$\bar{\mu}_i \equiv \sum_{r=1}^n d_{ir} T_r = 0$$

(assuming that all parameters are nonnegative and all numbers of targets T_r are integer), then either $d_{ir}=0$ or $T_r=0$ or both, for each r , i.e., type- i searchers can detect none of the targets present. In this case, the i^{th} term of the outer indicated product in (B.4.9) should be regarded as 1, as a target of any type that is present will certainly survive such a searcher.

Error bounds on the approximation (B.4.7) could probably be used in conjunction with the methods of Reference [9] to develop overall error bounds for the approximation (B.4.9) to the (unknown) exact T_j^K , but this has not been done here, and the resulting bounds might be large. It can be verified, however, that equation (B.4.9) reduces to the exact attrition equation in the special cases discussed above. Specifically:

- If the d_{ij} satisfy condition (B.2.1) then (B.4.9) reduces to the attrition equation defined by equations (B.2.6) and (B.2.7) of Section B.2.a (\bar{d}_i or zero case);
- If the d_{ij} satisfy condition (B.2.1) and all sets J_i are equal to $\{1, \dots, n\}$ then (B.4.9) reduces to equation (B.2.8) of Section B.2.b (\bar{d}_i only case);
- If $A=1$ (which implies that $a(j)=1$ and $U_{a(j)}=\{1, \dots, n\}$ for each j) then (B.4.9) reduces to equation (B.3.2) of Section B.3.a (ATRTAB Option 3 case); and
- If the d_{ij} satisfy condition (B.3.4), then (B.4.9) reduces to equation (B.3.9) of Section B.3.b (ATRTAB Option 2 case).

If the "no-parking-areas" conditions (B.1.1) hold, (B.4.9) reduces to the expression

$$T_j^K \approx T_j \left[1 - \prod_{i=1}^m \left(1 - \frac{k_{ij} d_{ij}}{\bar{\mu}_i} \left[1 - \prod_{r=1}^n (1 - d_{ir}) T_r \right] \right)^{S_i} \right], \quad (B.4.10)$$

where $\bar{\mu}_i$ is as defined in (B.4.5). This expression provides an approximate attrition equation in the no-parking-areas case with general d_{ij} ; it is an alternative to equation (17) of Reference [9]. Reference [14] provides additional discussion of this equation.

To repeat, Section B has explored combat processes with "uniform allocation of fire," where each searcher chooses its primary target uniformly from the targets it has detected. Section C examines a combat process with a different rule for target choice.

C. PARKING AREAS AND STRICT PRIORITY ALLOCATION OF FIRE

1. Introduction and Assumptions

Let the parameters n , m , T_j , S_i , U_a , d_{ij} , and k_{ij} be the same as in Section B.1.a. Suppose that there are m permutation mappings ψ_1, \dots, ψ_m , (one for each searcher type), each operating on the set of target types $\{1, \dots, n\}$. These mappings are to be interpreted as "targets of type $\psi_i(t)$ are the t^{th} priority for searchers of type i ," where lower t correspond to higher priority. (E.g., targets of type $\psi_2(1)$ are the highest priority targets for type-2 searchers.) With the parameters as just described, consider a combat process that proceeds according to Assumptions (1), (2), (3), (4), and (6) of Section B.1.a, but where Assumption (5) is replaced by Assumption

- (5') A searcher of type i that detects one or more targets chooses exactly one of these targets in such a way that the chosen target belongs to the highest priority type of the targets actually detected by that searcher. If more than one target of the highest priority type is detected, one target is chosen randomly and uniformly from among all those of that type detected. The searcher makes an attack on the parking area that contains the chosen target. (A searcher that makes no detections makes no attack.)

Assumption (5') will be called the "strict priority allocation of fire rule." It is "strict" in the sense that a searcher cannot be indifferent, regarding choice of primary target, between targets of different types--as reflected in the fact that the ψ_i are permutation mappings. It is also "strict" in that a searcher can never fire at a lower priority target if it has detected a higher priority one, even though, in reality, particular lower priority targets may occasionally draw fire away from higher priority ones because the lower priority ones are occupying particularly valuable territory, or for other reasons. (This rule is discussed more fully in Section B of Reference [12].)

To further illustrate the meaning of the ψ_i , consider the following example, adapted from Reference [8]. Suppose that $n=5$ and that for searchers of a given type i , we have $\psi_i = (3, 1, 5, 4, 2)$. That is, targets of type 3 are of the highest priority, followed by targets of type 1, and so on, with targets of type 2 of lowest priority. If the numbers of detected targets by a specific searcher of type i are 0, 5, 0, 8, 4 for targets of types 1, ..., 5, respectively, then the highest priority targets detected are of type 5 and (conditional on these numbers of detections) each of the four detected targets of type 5 will be chosen with probability $1/4$ to be the "primary target" of that particular searcher's attack.

Let T_j^K denote the expected number of type-j targets killed. It is perhaps paradoxical that an exact and algebraically tractable expression for T_j^K can be derived in the strict priority allocation of fire case with general d_{ij} , even though it was not possible to derive such an expression in the "simpler" uniform allocation of fire case. An (exact) attrition equation for the no-parking-areas case and strict priority allocation of fire has been developed in Reference [8]; Section C.2 below does the same for the parking areas case. The first attrition option of Subroutine ATRTAB is a straightforward special case of the strict priority allocation of fire process. To complete the presentation of the ATRTAB options, this special case is described in detail in a separate section (C.3).

2. Derivation of Attrition Equation

Assumption (4) and the appropriate methods of Reference [11] hold in the "priority" as well as the "uniform" allocation of fire process, thus Lemma 1 still holds, i.e.,

$$T_j^K = T_j \left[1 - \prod_{i=1}^m (1 - H_{ij})^{S_i} \right],$$

where H_{ij} is the probability that a specific type-i searcher kills a specific type-j target. Therefore consider specific searcher σ of type i and specific target τ of type j, which is located on parking area α , of type $a(j)$. Let the events K , F_α , G_a , and D be as defined in Section B.1.b, namely:

K -- σ kills τ (so $P(K)=H_{ij}$),

F_α -- σ attacks parking area α ,

G_a -- σ attacks a parking area of type a , and

D -- σ detects at least one target.

In addition, for each target type $r=1, \dots, n$ define the event

D_r --searcher σ detects at least one target of type r and no targets the types of which are of higher priority (for searchers of type i) than type r .

It is evident that

$$D = \bigcup_{r=1}^n D_r \quad (C.2.1)$$

and that this union is disjoint. If D_r occurs, then the "primary target" of σ 's attack will be of type r . The sets U_a partition $\{1, \dots, n\}$, i.e., no type of target can be located on more than

one type of parking area. Also, type- r targets are located on the (unique) type $a(r)$ of parking area. Thus

$$D_r \subset G_{a(r)}$$

and

$$D_r \cap G_{a'} = \emptyset \quad \text{for } a' \neq a(r).$$

Recall that the specific target τ being considered is of type j ; by the definition of the function a , for any target type r , $a(r) = a(j)$ precisely when $r \in U_{a(j)}$. Thus

$$\begin{aligned} P(G_{a(j)}) &= P(G_{a(j)} | \bar{D}) P(\bar{D}) + \sum_{r=1}^n P(G_{a(j)} | D_r) P(D_r) \\ &= \sum_{r \in U_{a(j)}} P(D_r). \end{aligned} \quad (C.2.2)$$

Equation (B.1.6) holds here, since different parking areas of the same type are assumed to be identical, i.e.,

$$P(F_\alpha | G_{a(j)}) = 1/M_{a(j)} \quad (C.2.3)$$

(and $P(F_\alpha | G_{a'}) = 0$ for $a' \neq a(j)$), and because of Assumption (6), equation (B.1.3), i.e.,

$$H_{ij} \equiv P(K) = k_{ij} P(F_\alpha) \quad (C.2.4)$$

also holds here.

Note that the priority ranking Ψ_i is a one-to-one function from $\{1, \dots, n\}$ onto itself and that its inverse Ψ_i^{-1} has the interpretation that $\Psi_i^{-1}(r)$ is the priority, for searchers of type i , of targets of type r . Therefore, by Assumptions (2) and (3),

$$P(D_r) = \left[1 - (1 - d_u)^{T_r} \right] \prod_{t=1}^{\Psi_i^{-1}(r)-1} \left(1 - d_{i, \Psi_i(t)} \right)^{T_{\Psi_i(t)}} \quad (C.2.5)$$

For those pairs (i, r) where $r = \Psi_i(1)$, so $\Psi_i^{-1}(r) = 1$, the indicated product in (C.2.5) runs from 1 to 0 and is thus meaningless. In this case, however, type- r targets are the highest

priority for type-i searchers, thus D_r occurs precisely when searcher σ (of type i) detects at least one type-r target. Thus if $r = \psi_i(1)$,

$$P(D_r) = 1 - (1-d_r)^{T_r}.$$

Expression (C.2.5) is therefore reasonable in all cases if the indicated product is interpreted as unity when $\psi_i^{-1}(r) = 1$.

It is straightforward to verify that $\sum_{r=1}^n P(D_r) = P(D)$, where $P(D)$ is as defined in (B.1.4).

Combining expressions (C.2.5), (C.2.2), (C.2.3), (C.2.4), and Lemma 1 yields the exact attrition equation

$$T_j^K = T_j \left[1 - \prod_{i=1}^m \left(1 - \frac{k_{ij}}{M_{a(j)}} \sum_{r \in U_{a(j)}} \left[[1 - (1-d_r)^{T_r}] \left[\prod_{v=1}^{\psi_i^{-1}(r)-1} (1-d_{i,\psi_i(v)})^{T_{\psi_i(v)}} \right] \right] \right)^{S_i} \right] \quad (C.2.6)$$

It can be verified that if the "no parking areas" conditions (B.1.1) hold, then equation (C.2.6) reduces to the equation given in the "Proposition" of Reference [8].

The reasoning of Reference [8] is somewhat different from the arguments here; Reference [8] does not use a symmetry argument such as equation (C.2.3) (which is used in the no-parking-areas case in Reference [12]) but conditions on the total number of targets searcher σ detects (cf. Reference [11]).

The strict priority allocation of fire rule may or may not be more realistic than the uniform allocation of fire rule, and some other rule may be more realistic than either of these; see, for example, the discussion in Reference [12].

3. A Specific Case--ATRTAB Option 1

The first option for computing attrition in Subroutine ATRTAB is a simple special case of equation (C.2.6). Only two types of targets are considered: open (nonsheltered) aircraft and aircraft shelters. The searchers are enemy aircraft attacking the airbase; the number of types of searchers can be a generic m . Open aircraft may be located on parking

areas with other open aircraft, but an attack on an aircraft shelter can kill only that shelter. The attack protocol is: if a searcher detects any open aircraft, it picks one at random (uniformly) from those it has detected and fires at its parking area; if a searcher detects no open aircraft but some aircraft shelters, it picks a shelter at random from those it has detected and fires at it.

In the notation of this paper, this situation can be considered a special case of the strict priority allocation of fire process, with the following conditions on the parameters:

$$\left. \begin{array}{l} n=2 \\ A=2 \\ U_a=\{a\} \quad \text{for } a=1,2 \\ \quad \quad \quad \text{(thus } a(j)=j \text{ for } j=1,2) \\ M_2=T_2 \\ \psi_i(j)=j \text{ for } j=1,2 \text{ and } i=1,\dots,m. \end{array} \right\} \quad (C.3.1)$$

That is, open aircraft are considered the type-1 targets and aircraft shelters, the type-2 targets. Substituting into equation (C.2.6) for $j=1$ and $j=2$, respectively, yields the expected number of open aircraft killed,

$$T_1^K = T_1 \left[1 - \prod_{i=1}^m \left(1 - \frac{k_{i1}}{M_1} \left[1 - (1-d_{i1}) T_1 \right] \right)^{S_i} \right], \quad (C.3.2)$$

and the expected number of aircraft shelters killed,

$$T_2^K = T_2 \left[1 - \prod_{i=1}^m \left(1 - \frac{k_{i2}}{T_2} \left[1 - (1-d_{i2}) T_2 \right] \left[(1-d_{i1}) T_1 \right] \right)^{S_i} \right]. \quad (C.3.3)$$

D. PROCESSES IN WHICH A SEARCHER DETECTS PARKING AREAS

In the combat processes examined in Sections B and C, the object of a searcher's attack is a parking area, but the searcher detects targets individually, and the parking area a searcher attacks is chosen via a random choice of detected target. This section explores an analogous but different set of combat processes, in which a searcher detects parking areas, and chooses one parking area to attack from those it has detected; every target on the

attacked parking area can be killed. These processes are thus closer to an "area fire" model. A series of exact and approximate expressions for the expected number of targets killed can be derived; these expressions parallel the series of attrition equations derived in Sections B and C, but have somewhat different forms. These expressions appear in Sections D.2 and D.3; Section D.1 presents the appropriate preliminaries. For conciseness, proof and derivation details have been omitted; they are similar to the methods of Sections B and C above, and of References [5], [9], and [11].

The parallels between the equations presented here and those derived in Sections B and C are heightened by the fact that if the parameters satisfy the "no-parking-areas" conditions (B.1.1), then each equation in Section B or C and its corresponding equation in Section D.2 or D.3 reduce to the same "no-parking-areas" equation. This, of course, follows from the fact that the underlying combat processes become identical. Section D.4 presents the details.

1. Terminology, Assumptions, and Proof Elements

Let the following input parameters be the same as in Section B.1.a:

- m = number of types of searchers,
- n = number of types of targets,
- A = number of types of parking areas,
- S_i = number of searchers of type i ($i=1,\dots,m$),
- T_j = number of targets of type j ($j=1,\dots,n$),
- M_a = number of parking areas of type a ($a=1,\dots,A$), and
- U_a = the subset of $\{1,\dots,n\}$ that gives the types of targets that are located on type- a parking areas.

It is (still) assumed that the sets U_a partition the set of target types $\{1,\dots,n\}$, and that all parking areas of a given parking area type contain identical complements of targets. (Some of the sets U_a could be empty, corresponding to parking areas that contain no targets; it might be realistic to use this option in some scenarios.) Thus the following parameters are also the same as in Section B.1.a:

- $a(j)$ = the (unique) type of parking area on which type- j targets are located,
 - $M_{a(j)}$ = the number of parking areas that can accommodate type- j targets,
- and
- t_j = $T_j/M_{a(j)}$ = the number of type- j targets on each type- $a(j)$ parking area.

It is necessary to assume that, for all j , $M_{a(j)} \geq 1$ if $T_j > 0$. Strictly speaking, the derivations are valid only if all the S_i , M_a , and t_j are nonnegative integers, but the resulting formulas are sensible with noninteger values of these parameters if all nonzero M_a and t_j are greater than or equal to unity (see Section B.1.a, above).

With the parameters as just described, consider a combat process that proceeds according to the following assumptions.

- (1) At a fixed time, all parking areas (and the targets located on them) become vulnerable to detection and attack.
- (2) Any particular searcher of type i detects any particular parking area of type a with probability q_{ia} .
- (3) Detections of different parking areas by a given searcher are mutually independent events.
- (4) The detection and attack processes of different searchers are mutually independent.
- (5) Of the parking areas it has detected, each searcher chooses one parking area according to a uniform distribution, and makes an attack on that parking area. (A searcher that makes no detections makes no attack.)
- (6) If an attack by a type- i searcher is made on a given parking area, then each type- j target located on that parking area is killed with probability k_{ij} . The effects of different attacks on the same parking area are independent.

As indicated earlier, these assumptions differ from those of Section B.1.a in that each searcher detects and attacks parking areas, rather than individual targets. It is still desired, however, to compute (for each j) the expected number of type- j targets (not parking areas) killed. Let T_j^K denote this quantity. Since Assumptions (4) and (6) are the same as before, Lemma 1 still holds, i.e.,

$$T_j^K = T_j \left[1 - \prod_{i=1}^m (1 - H_{ij})^{S_i} \right], \quad (D.1.1)$$

where H_{ij} is the probability that a specific type- i searcher kills a specific type- j target (which might also be killed by other searchers). Consider a specific searcher σ , of type i , and a specific target τ , of type j , which is located on parking area α (which is thus of type $a(j)$). Equation (B.1.3) continues to hold here, i.e.,

$$H_{ij} = k_{ij} P(F_\alpha) \quad (D.1.2)$$

where F_α is the event that searcher σ attacks parking area α . Furthermore, by Assumptions (2), (3), and (5),

$$P(F_\alpha) = \sum_{x=0}^{\infty} \frac{q_{i,a(j)}}{x+1} P(X=x), \quad (D.1.3)$$

where the random variable X represents the number of parking areas, other than parking area α , that searcher σ detects. (Equation (D.1.3) is comparable to equation (B.4.1).) The attrition equations in the next section are all derived by finding exact or approximate expressions for the probability distribution of X and applying equations (D.1.3), (D.1.2), and (D.1.1).

Assumption (5) could conceivably imply the attack of an empty parking area--if for some a and i , $M_a > 0$ and $q_{ia} > 0$, but $T_j = 0$ for all $j \in U_a$. This situation can be avoided, if desired, by preprocessing the data so that M_a is set equal to zero for all parking area types a where $T_j = 0$ for all $j \in U_a$ (so that, in effect, no parking areas are empty). (This preprocessing can also be performed, if desired, when the "strict priority allocation of fire" process of Section D.3, below, is used.)

2. Equations for the Uniform Allocation of Fire Cases

All the equations in this section follow from Assumptions (1) through (6) above, with additional assumptions as indicated.

a. Nonzero Detection Probabilities a Function of Searcher Type Only

Suppose that the detection probabilities q_{ia} satisfy the following property:

$$\left. \begin{array}{l} \text{For each } i, \text{ there exists a value } \bar{q}_i \in [0,1] \\ \text{and a subset } Q_i \text{ of } \{1, \dots, A\} \text{ such that} \\ \text{--for all } a \in Q_i, q_{ia} = \bar{q}_i \text{ and} \\ \text{--for all } a \notin Q_i, q_{ia} = 0. \end{array} \right\} \quad (D.2.1)$$

This is analogous to condition (B.2.1); some or all of the sets Q_i could be empty. In this case, it can be proved that the expected number of type- j targets killed is given by

$$T_j^K = T_j \left[1 - \prod_{i \in R_j} \left(1 - \frac{k_{ij}}{\bar{W}_i} \left[1 - (1 - \bar{q}_i)^{\bar{W}_i} \right] \right)^{S_i} \right] \quad (D.2.2)$$

where

$$\bar{W}_i = \sum_{a \in Q_i} M_a \quad i = 1, \dots, m, \quad (D.2.3)$$

and

$$R_j = \{i \mid a(j) \in Q_i \text{ and } \bar{W}_i > 0\}. \quad (D.2.4)$$

This equation is (in some sense) analogous to equations (B.2.6) and (B.2.7). An analogy to ATRTAB Option 2 can be developed as a special case of (D.2.2). In ATRTAB Option 2, searchers of a given type can only (detect and) attack one type of parking area. In the context of the process where searchers detect whole parking areas, this feature corresponds to each set Q_i having exactly one element, i.e., for each i , q_{ia} is nonzero for (at most) one value of a .

b. All Detection Probabilities a Function of Searcher Type Only

If the detection probabilities q_{ia} satisfy condition (D.2.1) and in addition, each set Q_i is equal to the whole set $\{1, \dots, A\}$ of parking area types, then detection probabilities are a function of searcher type only. In this case, the expected number of targets killed is given by

$$T_j^K = T_j \left[1 - \prod_{i=1}^m \left(1 - \frac{k_{ij}}{M} [1 - (1 - \bar{q}_i)^M] \right)^{S_i} \right], \quad (D.2.5)$$

where

$$M = \sum_{a=1}^A M_a \quad (D.2.6)$$

denotes the total number of parking areas. Equation (D.2.5) is analogous to equation (B.2.8). The special case of Equation (D.2.5) when $A=1$ can be considered as being analogous to the ATRTAB Option 3 equation (B.3.2).

c. Approximate Attrition Equations With General Detection Probabilities

In analogy with Section B.4, this section presents three expressions which represent reasonable approximations, if not exact values, for the expected number of targets

killed, T_j^K . These expressions accept general detection probabilities q_{ia} . Consider a specific searcher σ , of type i , and a specific target τ , of type j , which is located on parking area α , of type $a(j)$. The first expression utilizes the fact that X , the number of parking areas other than area α that searcher σ detects, is approximately Poisson distributed with mean

$$v_{ij} = q_{i,a(j)}(M_{a(j)} - 1) + \sum_{b \neq a(j)} q_{ib} M_b. \quad (D.2.7)$$

Applying equations (D.1.3), (D.1.2), and (D.1.1) yields

$$T_j^K \approx T_j \left[1 - \prod_{i=1}^m \left(1 - \frac{k_{ij} q_{i,a(j)}}{v_{ij}} [1 - e^{-v_{ij}}] \right)^{S_i} \right] \quad (D.2.8)$$

This approximation stands in analogy to expression (B.4.4), and as there, if $v_{ij} = 0$, $(1 - e^{-v_{ij}})/v_{ij}$ should be interpreted as unity.

Replacing v_{ij} in equation (D.2.8) with

$$\bar{v}_i = \sum_{b=1}^A q_{ib} M_b \quad (D.2.9)$$

yields the approximation

$$T_j^K \approx T_j \left[1 - \prod_{i=1}^m \left(1 - \frac{k_{ij} q_{i,a(j)}}{\bar{v}_i} [1 - e^{-\bar{v}_i}] \right)^{S_i} \right], \quad (D.2.10)$$

which is analogous to expression (B.4.6). As before, if $\bar{v}_i = 0$, $(1 - e^{-\bar{v}_i})/\bar{v}_i$ should be interpreted as unity. (If the condition holds that $M_{a(j)} > 0$ whenever $T_j > 0$, then $T_j > 0$ and $\bar{v}_i = 0$ implies that $q_{i,a(j)} = 0$, and the i th term of the indicated product is unity.)

Using the approximation

$$e^{-\bar{v}_i} = \prod_{b=1}^A (1 - q_{ib})^{M_b} \quad (D.2.11)$$

and substituting in (D.2.8) yields

$$T_j^K \approx T_j \left[1 - \prod_{i=1}^m \left(1 - \frac{k_{ij} q_{i,a(j)}}{\bar{v}_i} \left[1 - \prod_{b=1}^A (1 - q_{ib})^{M_b} \right]^{S_i} \right) \right], \quad (D.2.12)$$

which is analogous to expression (B.4.9). (If $\bar{v}_i=0$, regard the i^{th} term of the outer indicated product as unity.) It can be verified that (D.2.12) reduces to (D.2.2) or (D.2.5) if the appropriate conditions ((D.2.1) or (D.2.1) plus the added condition that all $Q_i = \{1, \dots, A\}$, respectively) hold.

3. Equation for the Strict Priority Allocation of Fire Case

In analogy to Section C, one can consider a combat process where a searcher detects parking areas and chooses to attack a parking area of the highest priority type it has detected. Let there be m permutation mappings ϕ_i (one for each searcher type), operating on the set $\{1, \dots, A\}$ of parking area types, to be interpreted as "parking areas of type $\phi_i(t)$ are the t^{th} priority for searchers of type i ." As in Section C, lower t correspond to higher priority. Let the parameters m , n , A , S_i , T_j , M_a , U_a , $a(j)$, q_{ia} , and k_{ij} be the same as in Section D.1; consider a combat process that proceeds according to Assumptions (1), (2), (3), (4), and (6) of Section D.1, but where Assumption (5) is replaced by Assumption

- (5'): A searcher of type i that detects one or more parking areas chooses exactly one of these parking areas in such a way that the chosen parking area belongs to the highest priority type of the parking areas actually detected by that searcher. If more than one parking area of the highest priority type is detected, one parking area is chosen randomly and uniformly from among all those of that type detected. The searcher makes an attack on the chosen parking area. (A searcher that makes no detections makes no attack.)

It can be shown that an exact expression for the expected number of type- j targets killed in this combat process is

$$T_j^K = T_j \left[1 - \prod_{i=1}^m \left(1 - \frac{k_{ij}}{M_{a(j)}} \left[1 - (1 - q_{i,a(j)})^{M_{a(j)}} \right]^{\phi_i^{-1}(a(j))-1} \prod_{v=1}^{\phi_i^{-1}(a(j))-1} (1 - q_{i,\phi_i(v)})^{M_{\phi_i(v)}} \right)^{S_i} \right] \quad (D.3.1)$$

As in Section C, for those pairs (i, j) where $a(j) = \phi_i(1)$, the inner indicated product is to be interpreted as unity.

If equation (D.3.1) is implemented in a combat model, then the quantities used by the equation can be computed in the model, based on other inputs--they need not be direct inputs to the model. In particular, the priority orderings ϕ_i can be computed by ranking the parking area types according to some desired measure of value, which can perhaps be a function of the numbers and types of targets on these areas. The effect of this might be that a searcher attacks (from among those parking areas it detects) the parking area with the most targets on it, or the parking area with a maximal weighted number of targets (with input weights that give relative values for targets of different types), or the parking area that contains the highest priority targets (as determined by some function ψ for target priority, cf. Section C, above).

4. Reduction to the "No-Parking-Areas" Case

Suppose the parameters satisfy the following conditions (which are identical to the conditions (B.1.1) of Section B):

$$\left. \begin{aligned} A &= n \\ a(j) &= j && \text{for } j = 1, \dots, n, \\ U_{a(j)} &= \{j\} && \text{for } j = 1, \dots, n, \text{ and} \\ M_{a(j)} &= T_j && \text{for } j = 1, \dots, n. \end{aligned} \right\} \quad (D.4.1)$$

Then each parking area contains exactly one target, and the combat processes of Sections D.1 and D.3 become identical to the "basic" combat processes, without parking areas, of References [5] (Chapter III) and [8], respectively. So do the processes of Sections B and C, as was indicated in those sections. One would thus expect corresponding attrition equations to reduce to the same "no-parking-areas" equation, and this indeed does occur.

For example, the expression for T_j^K given by (B.2.6) and (B.2.7) appears quite different from expression (D.2.2). Yet if the conditions (D.4.1) hold (and the appropriate

notation changes are made) both expressions for T_j^K reduce to the same equation. This "reduced" equation appears as equation (12) of Reference [11] and equation (21) of Reference [9]; it can compute attrition in the case where targets are not considered as being located on parking areas and where detection probabilities, if not zero, are a function only of searcher type.

The situation is similar for the other equations. Specifically, if the conditions (D.4.1) (i.e., (B.1.1)) hold, then:

1. (B.2.6)/(B.2.7) and (D.2.2) both reduce to equation (21) of Reference [9] (\bar{d}_i or 0 case, as stated above);
- 1a. If the condition (B.3.4) also holds, then (B.3.9) and the special case of (D.2.2) where each set Q_i contains one element both reduce to the special case of equation (21) of Reference [9] where each set J_i has exactly one element, i.e., each type of searcher can detect and attack only one type of target (ATRTAB Option 2 case);
2. (B.2.8) and (D.2.5) both reduce to equation (3.14) of Reference [6], which also appears in several other references, as indicated in Section B.2.b, above (\bar{d}_i only case);
- 2a. (B.3.2) with $n=1$ and the special case of (D.2.5) where $A=1$, so that $\bar{q}_i=q_{i1}$ for all i , both reduce to the special case of equation (3.14) of Reference [6] where there is only one type of target (ATRTAB Option 3 case);
3. (B.4.4) and (D.2.8) both reduce to equation (13) of Reference [9] (first Poisson approximation case);
4. (B.4.6) and (D.2.10) both reduce to equation (14) of Reference [9] (second Poisson approximation case);
5. (B.4.9) and (D.2.12) both reduce to equation (B.4.10) of the current paper, which is also equation (B.3.1) of Reference [14] (general d_{ij} binomial approximation); and
6. (C.2.6) and (D.3.1) both reduce to the equation given in the "Proposition" of Reference [8] (strict priority allocation of fire case).

E. CONCLUSIONS AND SUGGESTIONS FOR FURTHER WORK

This paper has derived a number of attrition equations that treat the situation where an attack on one target can kill other targets located in the same area, under varying assumptions about the detection and attack processes. The work has been related to 1) a procedure used in several combat simulations to model attrition caused by airbase attack

and 2) several attrition equations derived for the no-parking-areas case. This section presents some possible extensions of the work on attrition processes with parking areas.

A series of different combat processes can be generated by varying the specification of targets killed on a parking area, given attack; one could then try to find formulas for the expected numbers of targets killed for these processes. One possibility is:

1. N targets (if there are that many) are selected at random from the parking area that a searcher attacks, and only those targets are vulnerable to that attack, where N is an input (possibly unity).

The parking area that a searcher attacks could be chosen according to (the first five assumptions of) any one of the processes of Section B, C, or D (or some other process).

Some other variations, which involve the idea of "primary target" (the detected one that was "chosen"), and are thus not (directly) applicable to the combat process of Section D, are as follows.

2. A searcher attacks its primary target with some input probability; with one minus that probability, the searcher attacks some other target (where the choice of target is made in some specified manner) on the same parking area as the primary target. Only the attacked target can be killed; the probability of kill given attack can depend on whether the attacked target is the primary target or not.
3. Every target on the attacked parking area is vulnerable, but the primary target is killed with higher (input) probability.
4. The primary target and exactly N-1 other targets (chosen in some specified manner) on the attacked parking area are vulnerable; the probability of kill given attack may be different for the primary target (N is an input).

Item 4 is in some sense a combination of Items 1 and 3.

Some other possible topics for future work are as follows.

5. Develop an attrition equation for the case where target choice is based on a "nonstrict priority allocation of fire" rule, in which an attacker might be indifferent between targets (or parking areas) of some different types.
6. Develop an attrition equation for the case of "weighted allocation of fire," as discussed in Reference [12], where target choice is based on a set of input weights for targets (or parking areas) of different types. (An exact, succinct expression for the expected number of targets killed might not be possible, but there could be approximate or heuristically reasonable expressions.) Related to this is

7. Adapt the "relatively general attrition equation" of Reference [1] to also treat the case of parking areas.
8. The derivations in this paper have frequently utilized the assumption that the sets U_a partition the set of target types $\{1, \dots, n\}$, so that each type of target is located on exactly one type of parking area. Develop attrition equations in the case where targets of a given type might be located on several types of parking areas. (In the uniform allocation of fire case, targets of the same type located on different types of parking areas can simply be considered as targets of different types--with the same detection and kill probabilities--but this doesn't work under priority allocation of fire rules.)
9. Develop error bounds for the approximations (B.4.4), (B.4.6), (B.4.9) (D.2.8), (D.2.10), and (D.2.12) to the expected number of targets killed.

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